A Closed Contour of Integration in Regge Calculus

Danny Birmingham [1]

University College Dublin, Department of Mathematical Physics,
Belfield, Dublin 4, Ireland

Abstract

The analytic structure of the Regge action on a cone in d dimensions over a boundary of arbitrary topology is determined in simplicial minisuperspace. The minisuperspace is defined by the assignment of a single internal edge length to all 1-simplices emanating from the cone vertex, and a single boundary edge length to all 1-simplices lying on the boundary. The Regge action is analyzed in the space of complex edge lengths, and it is shown that there are three finite branch points in this complex plane. A closed contour of integration encircling the branch points is shown to yield a convergent real wave function. This closed contour can be deformed to a steepest descent contour for all sizes of the bounding universe. In general, the contour yields an oscillating wave function for universes of size greater than a critical value which depends on the topology of the bounding universe. For values less than the critical value the wave function exhibits exponential behaviour. It is shown that the critical value is positive for spherical topology in arbitrary dimensions. In three dimensions we compute the critical value for a boundary universe of arbitrary genus, while in four and five dimensions we study examples of product manifolds and connected sums.

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[1] Email: Dannyb@ollamh.ucd.ie

1 Introduction

Regge calculus provides a framework in which one can study a simplicial approach to the quantization of the gravitational field [1]-[5]. The basic idea is to model the spacetime of interest by a simplicial complex, in which the edge length assignments become the dynamical variables. A central question in any approach to such a problem is how to define the associated functional integral. In particular, one must deal with the well-known convergence problems of the corresponding continuum Euclidean functional integral [6].

In [7], the functional integral for a simplicial complex with a single boundary component of S^3 topology was studied. It was shown that by taking the dynamical variables to be given by the space of complex valued edge lengths, one could study the convergence and physical properties of the associated Hartle-Hawking wave function [8] in explicit detail. One simplifying feature introduced was to restrict attention to a simplicial minisuperspace. This minisuperspace consisted of the assignment of a single internal edge length variable, and a single boundary edge length variable to the simplicial complex. The functional integral then reduced to an integral over a single complex variable. In particular, it was shown that the Regge action had three square root branch points, and that a closed contour of integration encircling these points led to a convergent result. More particularly, this contour had the appealing feature that the form of the resulting wave function satisfied certain physical requirements. The general criteria for defining the wave function of the universe were examined in [9]. For values of the boundary edge length greater than a critical value, the wave function behaved in an oscillatory manner (corresponding to a classically allowed regime), and was dominated semiclassically by real simplicial geometries of Lorentzian signature. For values of the boundary edge length less than the critical value, the wave function behaved in an exponential way (corresponding to a classically forbidden regime), with a semiclassical domination by real geometries of Euclidean signature.

In [10, 11], this result was extended to a class of non-trivial topologies, given by the lens spaces $L(k, 1)$. The essential ingredient in this generalization was the observation that the features in [7] which allowed the analysis to proceed were the particular simplicial minisuperspace considered, and the fact that the spacetime simplicial complex was chosen to be a cone over S^3.

In this paper, we consider Regge calculus on a cone in arbitrary dimensions over a boundary of arbitrary topology. The basic object of interest is the corresponding functional integral which yields the Hartle-Hawking wave function for a universe with the boundary topology. For a simplicial complex with the structure of a cone, there is a single internal vertex, and we study the model for a simplicial minisuperspace in which all edge lengths which emanate from this cone vertex take equal value. In addition, we take all boundary edge lengths to be equal. As in the cases discussed above, we study the functional integral in the space of complex valued edge lengths. The basic result is that due to the cone structure and simplicial minisuperspace, the analytic structure of the Regge action parallels that studied in the four-dimensional case. In particular, there are three square root branch points, and consequently we can appeal to the general argument presented in [7]. We remark here that the simplicial cone structure is not a simplicial manifold, as defined in section three. Nevertheless, the formalism of Regge calculus is de-

fined in this more general setting. Our aim is to obtain the Hartle-Hawking wave function of the associated boundary universe, and study its physical properties for a wide variety of topologies. In particular, we evaluate the critical edge length value which allows us to determine the classically forbidden and allowed regimes.

The outline of this work is as follows. In the following section, we determine the analytic structure of the Regge action for a simplicial complex which is a cone over a boundary of arbitrary topology in d dimensions. We find that the action has three square root branch points, and the classical extrema of the action are obtained. It is shown that for boundary edge lengths less than a critical value, there are real simplicial geometries of Euclidean signature, and for boundary edge lengths greater than the critical value, there are classical extrema corresponding to real geometries of Lorentzian signature. The analytic structure of the action is shown to parallel that found in [7], and thus a closed contour of integration encircling all three branch points yields a convergent wave function, with the desired physical properties. Following this general analysis, we then study particular examples in specific dimensions. We also determine the form of the wave function for a universe with spherical topology in arbitrary dimensions.

2 The Regge Action on a Cone

Given a simplicial complex M_d with boundary $\partial M_d = M_{d-1}$, the wave function is defined by

$$\Psi(s_b) = \int_C d\mu(s_i)\ \exp[-S(s_b, s_i)]. \tag{1}$$

Here, the variables s_b specify the edge lengths of the boundary and the integration is over the internal dynamical variables s_i. The form of the measure μ, the contour of integration C, along with the action S, are required to complete the specification of the model.

The Euclidean Einstein action with cosmological term for a manifold with boundary is given by

$$S = -\frac{1}{16\pi G}\int_M d^d x\ \sqrt{g}R + \frac{2\Lambda}{16\pi G}\int_M d^d x\ \sqrt{g} - \frac{2}{16\pi G}\int_{\partial M} d^{(d-1)}x\ \sqrt{h}K, \tag{2}$$

where R is the scalar curvature of the metric g, Λ is the cosmological constant, and K is the extrinsic curvature scalar of the induced metric h on the boundary. The simplicial analogue of this action is the corresponding Regge action [1, 12], which is given by

$$\begin{aligned} S \;&=\; -\frac{2}{l^{d-2}}\sum_{\sigma_{d-2}\subset int(M_d)} V_{d-2}(\sigma_{d-2})\theta(\sigma_{d-2}) + \frac{2\Lambda}{l^{d-2}}\sum_{\sigma_d\subset int(M_d)} V_d(\sigma_d) \\ &\quad - \frac{2}{l^{d-2}}\sum_{\sigma_{d-2}\subset\partial M_d} V_{d-2}(\sigma_{d-2})\psi(\sigma_{d-2}), \end{aligned} \tag{3}$$

where the Planck length, in units where $\hbar = c = 1$, is $l = (16\pi G)^{1/(d-2)}$. The various terms in (3) are described as follows. The Einstein term is represented by a summation over internal $(d-2)$-simplices $\sigma_{d-2} \subset int(M_d)$ (also known as hinges). An internal hinge is any $(d-2)$-simplex of the complex which contains at least one internal vertex, and the notation $int(M_d)$ is used to denote this set. The form of the Einstein action involves the

volume of the hinge $V_{d-2}(\sigma_{d-2})$ and the associated deficit angle $\theta(\sigma_{d-2})$. Similarly, the boundary term is given in terms of the boundary $(d-2)$-simplices and their associated deficit angles denoted by $\psi(\sigma_{d-2})$. The cosmological term is simply represented as a sum over the volumes $V_d(\sigma_d)$ of the d-simplices σ_d of the complex.

The deficit angle of an internal $(d-2)$-simplex σ_{d-2}^{int} is given by

$$\theta(\sigma_{d-2}^{int}) = 2\pi - \sum_{\sigma_d \supset \sigma_{d-2}^{int}} \theta_{dih}(\sigma_{d-2}^{int}, \sigma_d) \quad , \tag{4}$$

where the summation is over the dihedral angles of all d-simplices containing σ_{d-2}^{int}. The deficit angle for the boundary $(d-2)$-simplex σ_{d-2}^{bdy} is

$$\psi(\sigma_{d-2}^{bdy}) = \pi - \sum_{\sigma_d \supset \sigma_{d-2}^{bdy}} \theta_{dih}(\sigma_{d-2}^{bdy}, \sigma_d) \quad . \tag{5}$$

A simple way to obtain a simplicial complex with a boundary is to take the simplicial complex to be the cone over the boundary [13]. The simplicial cones considered here belong to the general class of simplicial conifolds discussed in [14]. Given a $(d-1)$-dimensional simplicial complex M_{d-1}, the cone over M_{d-1} is denoted by $M_d = c \star M_{d-1}$ and is described as follows [13]. One takes the vertex set of M_{d-1} and adds the cone vertex c. The cone vertex is then joined to each of the vertices of M_{d-1}. Each d-simplex of the cone is then of the form $[c, \sigma_{d-1}]$, where σ_{d-1} is a $(d-1)$-simplex of the boundary complex. With this orientation, the boundary of the cone is $\partial(c \star M_{d-1}) = M_{d-1}$. In fact, the number of simplices of each dimension is given by

$$\begin{aligned} N_0(M_d) &= N_0(M_{d-1}) + 1, \\ N_1(M_d) &= N_1(M_{d-1}) + N_0(M_{d-1}), \\ &\vdots \\ N_{d-2}(M_d) &= N_{d-2}(M_{d-1}) + N_{d-3}(M_{d-1}), \\ N_{d-1}(M_d) &= N_{d-1}(M_{d-1}) + N_{d-2}(M_{d-1}), \\ N_d(M_d) &= N_{d-1}(M_{d-1}). \end{aligned} \tag{6}$$

Thus, we see that there is a total of $N_{d-2}(M_{d-1})$ boundary $(d-2)$-simplices, and a total of $N_{d-3}(M_{d-1})$ internal $(d-2)$-simplices.

So let us consider a d-simplex of this cone given by $\sigma_d = [c, 1, 2, ..., d]$. We let $e_1, e_2, ..., e_d$ be 1-forms associated with the 1-simplices $[c, 1], ..., [c, d]$. The volume d-form for σ_d is then given by $\omega_d = e_1 \wedge ... \wedge e_d$. The volume of σ_d is

$$V_d(\sigma_d) = \frac{1}{d!} Det^{\frac{1}{2}}(M_d), \tag{7}$$

where $M_d \equiv \omega_d \cdot \omega_d$ has entries $m_{ij} = e_i \cdot e_j = \frac{1}{2}(s_{ci} + s_{cj} - s_{ij})$, and s_{ij} is the squared edge length assigned to the 1-simplex $[i, j]$, see for example [3].

In order to compute the internal deficit angles, we need a formula for the dihedral angle of the internal $(d-2)$-simplex $\sigma_{d-2}^{int} = [c, 1, 2, ..., d-2]$ which is contained in σ_d.

This $(d-2)$-simplex is shared by the two $(d-1)$-simplices $\sigma_{d-1} = [c, 1, 2, ..., d-2, d-1]$ and $\sigma'_{d-1} = [c, 1, 2, ..., d-2, d]$. The volume forms of σ_{d-1} and σ'_{d-1} are

$$\begin{aligned} \omega_{d-1} &= e_1 \wedge ... \wedge e_{d-2} \wedge e_{d-1}, \\ \omega'_{d-1} &= e_1 \wedge ... \wedge e_{d-2} \wedge e_d. \end{aligned} \tag{8}$$

The dihedral angle is

$$\cos\theta_{dih}(\sigma^{int}_{d-2}, \sigma_d) = \frac{1}{((d-1)!)^2} \frac{Det(M_{d-1})}{V_{d-1}(\sigma_{d-1}) V_{d-1}(\sigma'_{d-1})}, \tag{9}$$

where $M_{d-1} = \omega_{d-1} \cdot \omega'_{d-1}$.

For the boundary dihedral angle, we consider the $(d-2)$-simplex $\sigma^{bdy}_{d-2} = [1, 2, ..., d-1]$. We choose an orientation such that the vertices $1, 2, ..., d-1$ appear at the beginning of the two $(d-1)$-simplices as $\sigma_{d-1} = [1, 2, ..., d-1, c]$ and $\sigma'_{d-1} = [1, 2, ..., d-1, d]$. The volume forms of σ_{d-1} and σ'_{d-1} are

$$\begin{aligned} \omega_{d-1} &= (e_2 - e_1) \wedge (e_3 - e_1) \wedge ... \wedge (e_{d-1} - e_1) \wedge (-e_1), \\ \omega'_{d-1} &= (e_2 - e_1) \wedge (e_3 - e_1) \wedge ... \wedge (e_{d-1} - e_1) \wedge (e_d - e_1). \end{aligned} \tag{10}$$

The dihedral angle is then

$$\cos\theta_{dih}(\sigma^{bdy}_{d-2}, \sigma_d) = \frac{1}{((d-1)!)^2} \frac{Det(M_{d-1})}{V_{d-1}(\sigma_{d-1}) V'_{d-1}(\sigma_{d-1})}. \tag{11}$$

In order to obtain an explicit expression for the action of a d-dimensional cone, one needs to compute various volumes and deficit angles. The important point for our purposes is that we can obtain closed expressions for these in the simplicial minisuperspace of interest. We assign an edge length variable s_i to each of the internal 1-simplices, and an edge length variable s_b to each of the boundary 1-simplices. It is useful to define the dimensionless ratio $z = s_i/s_b$.

Let us begin by considering the volume of an internal n-simplex $\sigma^{int}_n = [c, 1, 2, ..., n]$. From (7), we see that it takes the form

$$V_n(\sigma^{int}_n) = \frac{s_b^{n/2}}{n!} Det^{\frac{1}{2}}(M_n), \tag{12}$$

where

$$M_n = \begin{pmatrix} z & z - \frac{1}{2} & \cdots & z - \frac{1}{2} \\ z - \frac{1}{2} & z & \cdots & z - \frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ z - \frac{1}{2} & z - \frac{1}{2} & \cdots & z \end{pmatrix}. \tag{13}$$

The determinant can be evaluated explicitly by row reducing M_n to echelon form. Thus, from (12), the volume of an internal n-simplex is given by

$$V_n(\sigma^{int}_n) = \frac{s_b^{n/2}}{n!} \sqrt{\frac{n}{2^{n-1}}} \sqrt{z - \frac{n-1}{2n}}. \tag{14}$$

In order to compute the volume of a boundary n-simplex $\sigma_n^{bdy} = [1, ..., n]$, we can proceed along similar lines. However, one notices that the boundary volumes can be obtained from (14) by setting $z = 1$. Thus, the volume of a boundary n-simplex is

$$V_n(\sigma_n^{bdy}) = \frac{s_b^{n/2}}{n!}\sqrt{\frac{n+1}{2^n}}. \tag{15}$$

The dihedral angles are computed as follows. Consider first the case of an internal $(d-2)$-simplex $\sigma_{d-2}^{int} = [c, 1, ..., d-2]$. From (9), we see that it is given by

$$\cos\theta_{dih}(\sigma_{d-2}^{int}, \sigma_d) = \frac{s_b^{d-1}}{((d-1)!)^2}\frac{Det(M_{d-1})}{V_{d-1}(\sigma_{d-1})V_{d-1}(\sigma'_{d-1})}, \tag{16}$$

where

$$M_{d-1} = \begin{pmatrix} z & z-\frac{1}{2} & \cdots & z-\frac{1}{2} \\ z-\frac{1}{2} & z & \cdots & z-\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ z-\frac{1}{2} & z-\frac{1}{2} & \cdots & z-\frac{1}{2} \end{pmatrix}. \tag{17}$$

In this case, one of the volume factors is of internal type and one is of boundary type. Thus, the dihedral angle of internal type is given by

$$\cos\theta_{dih}(\sigma_{d-2}^{int}, \sigma_d) = \frac{z-\frac{1}{2}}{(d-1)\left(z-\frac{d-2}{2(d-1)}\right)}. \tag{18}$$

Consider now a boundary $(d-2)$-simplex $\sigma_{d-2}^{bdy} = [1, 2, ..., d-1]$. This is shared by the two $(d-1)$-simplices contained in $\sigma_d = [c, 1, 2, ..., d]$, namely $\sigma_{d-1} = [c, 1, 2, ..., d-1]$ and $\sigma'_{d-1} = [1, 2, ..., d-1, d]$ which are of internal and boundary type, respectively. The boundary dihedral angle is then computed to be

$$\cos\theta_{dih}(\sigma_{d-2}^{bdy}, \sigma_d) = \frac{1}{\sqrt{2d(d-1)\left(z-\frac{d-2}{2(d-1)}\right)}}. \tag{19}$$

With these results at hand, it is straightforward to write down the complete analytic form of the action. It is convenient to write it in the form

$$S(z,R) = \frac{1}{H^{d-2}}\left[-R^{\frac{d}{2}-1}F(z) + R^{\frac{d}{2}}G(z)\right], \tag{20}$$

where

$$\begin{aligned} F(z) &= \frac{dN_{d-1}(M_{d-1})}{(d-2)!}\sqrt{\frac{d-1}{2^{d-2}}}\left[\pi - 2\arccos\left(\frac{1}{\sqrt{2d(d-1)(z-z_2)}}\right)\right] \\ &+ \frac{2N_{d-3}(M_{d-1})}{(d-2)!}\sqrt{\frac{d-2}{2^{d-3}}}\sqrt{z-z_1}\left[2\pi\right. \\ &- \left.\frac{d(d-1)N_{d-1}(M_{d-1})}{2N_{d-3}(M_{d-1})}\arccos\left(\frac{z-\frac{1}{2}}{(d-1)(z-z_2)}\right)\right], \\ G(z) &= \frac{6N_{d-1}(M_{d-1})}{d!}\sqrt{\frac{d}{2^{d-1}}}\sqrt{z-z_3}. \end{aligned} \tag{21}$$

In the above, we have introduced the dimensionless variable $R = H^2 s_b/l^2$, where $H^2 = l^2\Lambda/3$. It is also convenient to define

$$z_1 = \frac{d-3}{2(d-2)}, \quad z_2 = \frac{d-2}{2(d-1)}, \quad z_3 = \frac{d-1}{2d}. \tag{22}$$

We thus see that the action is specified completely in terms of the number $N_{d-1}(M_{d-1})$ of $(d-1)$-simplices, and the number $N_{d-3}(M_{d-1})$ of $(d-3)$-simplices of the boundary. It is in this sense that the form of the action depends only on simple data associated with the boundary.

It is important to make the following observation regarding the factor of 2 appearing in the formula for the deficit angle of a boundary $(d-2)$-simplex (the first term in $F(z)$). The bounding space M_{d-1} is represented by a closed simplicial complex, and closure of the complex means that each $(d-2)$-simplex is contained in precisely two $(d-1)$-simplices. Thus, when we elevate this boundary complex to its associated cone, we immediately know that the number of d-simplices containing each boundary $(d-2)$-simplex is again precisely 2, each being of the form $[c, \sigma_{(d-1)}]$, with $\sigma_{(d-1)}$ belonging to the boundary. This fact is crucially relevant when we search for the extrema of the action.

In computing the Einstein term, we note that each internal $(d-2)$-simplex is of the form $\sigma^{int}_{d-2} = [c, \sigma^{bdy}_{d-3}]$. The number of d-simplices containing σ^{int}_{d-2} is then equal to the number of boundary $(d-1)$-simplices containing σ^{bdy}_{d-3}. This number depends on the individual $(d-3)$-simplex. However, because of the structure of the simplicial minisuperspace, we only need the total number of boundary $(d-3)$-simplices contained in all the boundary $(d-1)$-simplices. Bearing in mind that each boundary $(d-1)$-simplex contains a total $d(d-1)/2$ of $(d-3)$-simplices, we obtain the Einstein term as given above.

The analytic structure of the action is immediately identified. We see that there are three finite square root branch points at z_1, z_2, z_3. It is also clear that it is respectively the vanishing of the internal $(d-2)$-volume, internal $(d-1)$-volume, and internal d-volume, which is responsible for the presence of these branch points. One should also note that there is a logarithmic branch point at z_2.

In order to be able to interpret the nature of the wave function, it is necessary to know the signature of the metric tensor for each of the d-simplices. This is obtained by finding the eigenvalues of the metric tensor, M_d of eqn. (7), of each d-simplex. Once again, this calculation can be performed by row reduction, yielding the result that there is one eigenvalue $\lambda = \frac{1}{2}$ with a multiplicity of $(d-1)$, and an eigenvalue $\lambda = d\,(z - z_3)$. Thus, we see that for real $z > z_3$, we have a regime of real Euclidean signature metrics, while for real $z < z_3$ we have a regime of real Lorentzian signature metrics.

Given the presence of the branch points, we must declare the location of the branch cuts. The function $\arccos(z)$ has branch points at $z = \pm 1, \infty$, and conventionally one places the branch cuts along the real axis from $-\infty$ to -1 and $+1$ to $+\infty$. The function $\arccos(z)$ is real for real z lying between -1 and $+1$. For the second arccos term in (21), we note that the corresponding branch cuts lie between the points z_2 to z_3, and z_1 to z_2. The first arccos term has a branch cut from z_2 to z_3. Furthermore, due to the presence of the square root branch points at z_1, z_2, z_3, we may declare a suitable branch cut for the total action as one which extends from z_1 to $-\infty$ along the real axis. We note the similarity between this case and that studied in [7] and [10]. The only difference in the

d-dimensional case under study is that the location of the three branch points depends on the dimension. With this convention, we note that for real $z > z_3$, we have real valued Euclidean signature action, with real volumes and real deficit angles. The action on the first sheet is then given by (20)-(21) with positive signs taken for the square root factors.

It should also be noted that the action is purely imaginary for real $z < z_1$. On the first sheet, we have

$$\begin{aligned} F(z) &= i\frac{dN_{d-1}(M_{d-1})}{(d-2)!}\sqrt{\frac{d-1}{2^{d-2}}}\left[-2\,\mathrm{arcsinh}\left(\frac{1}{\sqrt{2d(d-1)(z_2-z)}}\right)\right] \\ &+ \frac{2N_{d-3}(M_{d-1})}{(d-2)!}\sqrt{\frac{d-2}{2^{d-3}}}\sqrt{z_1-z}\left[2\pi\right. \\ &- \left.\frac{d(d-1)N_{d-1}(M_{d-1})}{2N_{d-3}(M_{d-1})}\arccos\left(\frac{z-\frac{1}{2}}{(d-1)(z-z_2)}\right)\right], \\ G(z) &= i\frac{6N_{d-1}(M_{d-1})}{d!}\sqrt{\frac{d}{2^{d-1}}}\sqrt{z_3-z}, \end{aligned} \tag{23}$$

where the identity,

$$\pi - 2\arccos(iz) = 2\arcsin(iz) = 2i\,\mathrm{arcsinh}(z), \tag{24}$$

has been used. It is at this point that we notice the relevance of the factor of 2 in the formula for the deficit angle of a boundary $(d-2)$-simplex. Its presence allows use of the formula (24), thereby rendering the action purely imaginary in the region real $z < z_1$. Thus, when the action is continued once around all three branch points, we reach a second sheet, and the value of the action is the negative of its value on the first sheet, as can be seen by using the identity $\arccos(-z) = \pi - \arccos(z)$. A continuation twice around all three branch points returns the action to its original value.

The asymptotic behaviour of the action on its various sheets is important for the determination of convergent contours of integration. For large $\mid z \mid$ on the first sheet, we have

$$S(z,R) \sim \frac{6N_{d-1}(M_{d-1})}{d!}\sqrt{\frac{d}{2^{d-1}}}\frac{R^{\frac{d}{2}-1}}{H^{d-2}}(R-R_{crit})\sqrt{z}, \tag{25}$$

where

$$R_{crit} = \frac{2d(d-1)N_{d-3}(M_{d-1})}{3N_{d-1}(M_{d-1})}\sqrt{\frac{d-2}{d}}\left[2\pi - \frac{d(d-1)N_{d-1}(M_{d-1})}{2N_{d-3}(M_{d-1})}\arccos\left(\frac{1}{d-1}\right)\right]. \tag{26}$$

Thus, we see that the asymptotic behaviour of the action depends crucially on whether the boundary edge length R is greater than or less that the critical value R_{crit}. It is also important to notice that R_{crit} depends only on the boundary data $N_{d-1}(M_{d-1})$ and $N_{d-3}(M_{d-1})$.

Turning now to a description of the classical extrema of the action, we have the Regge equation of motion given by

$$\frac{d}{dz}S(z,R) = 0. \tag{27}$$

This equation is to be solved for the value of z subject to fixed boundary data R. The internal lengths are then fixed by the relation $s_i = \frac{zRl^2}{H^2}$. We also impose the physical restriction that the boundary data R should be real valued and positive. It is straightforward to show that the Regge equation can be written as

$$R = \frac{F'(z)}{G'(z)}, \tag{28}$$

where the prime denotes a derivative with respect to z. It is thus clear that classical solutions can exist when both F' and G' are purely real (which is the case for real $z > z_3$), or when both F' and G' are purely imaginary (which is the case for real $z < z_1$). Using (26), we then determine that for every $0 < R < R_{crit}$, there is a real solution of Euclidean signature at every real $z > z_3$. Additionally, for every positive $R > R_{crit}$, there is a real solution of Lorentzian signature for every real $z < z_1$. These solutions occur in pairs, and by encircling all three branch points once, one obtains a second solution with opposite value for the action.

The main aim is to compute the wave function of the model. In general, this problem can be tackled by first determining the classical extrema of the action, and then searching for steepest descent contours of integration which yield a convergent result. In this approach, the contour depends on the boundary value R. In [7], this procedure was followed for a four-dimensional model with a universe of S^3 topology. However, it was also observed that the model possessed a closed contour of integration which encircled all three branch points, and led to a convergent wave function. Furthermore, this closed contour of integration could be deformed to the steepest descent contours for all values of the boundary edge length. Thus, the closed contour provides a contour specification which is independent of the argument of the wave function, namely the boundary edge length, and as such it constitutes a contour prescription for the model [15].

In [10, 11], this result was extended to non-trivial topology in four dimensions. Specifically, the case where the four-dimensional spacetime had the structure of a cone over a boundary of lens space topology was studied. It was shown that the Regge action once again had three square root branch points, and thus the analysis of [7] could be applied to the more general case. It was observed that the key features in this application were the simplicial minisuperspace consisting of a single internal edge and a single boundary edge, and the cone structure of the spacetime.

In the above, we have computed the Regge action for a cone over a boundary of arbitrary topology in d dimensions. It is clear that, due to the analytic and asymptotic structure of the action, the analysis of [7] also applies in this general setting. Indeed, we have seen that the action has three square root branch points. Furthermore, the fact that the action is purely real for real $z > z_3$ and purely imaginary for real $z < z_1$ ensures that the extrema of the action are also obtained in an analogous fashion. The essential point to be noted in the general case is that the branch points lie along the real z-axis at locations which depend on the particular dimension under study. In addition, the critical value of the boundary edge length is determined in terms of the number $N_{d-1}(M_{d-1})$ of $(d-1)$-simplices, and number $N_{d-3}(M_{d-1})$ of $(d-3)$-simplices of the boundary. The simplicity of this result allows to to quickly survey a large number of non-trivial topologies and determine the structure of the corresponding wave function.

3 Three Dimensions

Our aim in this and the following sections is to determine the behaviour of the wave function for universes of non-trivial topology in various dimensions. As we have seen, the nature of the wave function is dependent on the value of the critical length R_{crit}. Here, we shall compute the value of R_{crit} for a wide variety of models. For many of the models studied, we find that R_{crit} is negative, and thus the wave function oscillates for all positive values of the boundary edge length. In particular, we then find that the resulting wave function supports a classically allowed regime, whereby the semiclassical approximation is dominated by real Lorentzian signature geometries.

Before dealing specifically with the three-dimensional case, we make some general remarks. For the models under consideration, we are taking the boundary complex to be a simplicial manifold. Given an n-dimensional simplicial complex K, we recall that the star of a simplex σ in K is the collection of simplices which contain σ, together with all their subsimplices. The link of σ is then the set of simplices in the star of σ which do not contain σ. The simplicial complex K is said to be a simplicial manifold if and only if the link of every k-simplex is combinatorially equivalent to an $(n-k-1)$-sphere [16].

For a simplicial manifold, the following Dehn-Sommerville relations are satisfied [17]

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{d}(-1)^i N_i, \\ 0 &= \sum_{i=2k-1}^{d}(-1)^i \begin{pmatrix} i+1 \\ 2k-1 \end{pmatrix} N_i, \quad \text{if } d \text{ even}, \ 1 \leq k \leq \frac{d}{2}, \\ 0 &= \sum_{i=2k}^{d}(-1)^i \begin{pmatrix} i+1 \\ 2k \end{pmatrix} N_i, \quad \text{if } d \text{ odd}, \ 1 \leq k \leq \frac{d-1}{2}, \end{aligned} \tag{29}$$

where $\chi(M)$ is the Euler characteristic of M. We shall use these relations in the following.

In three dimensions, the most general possibility for the topology of the boundary universe is a genus g Riemann surface Σ_g. In this case, the critical length is given by

$$R_{crit} = \frac{4\pi}{\sqrt{3}} \frac{1}{N_2(\Sigma_g)} \left[2N_0(\Sigma_g) - N_2(\Sigma_g)\right]. \tag{30}$$

However, since Σ_g is a simplicial manifold, the Dehn-Sommerville relations state that

$$\begin{aligned} N_0(\Sigma_g) - N_1(\Sigma_g) + N_2(\Sigma_g) &= \chi(\Sigma_g) = (2-2g), \\ 2N_1(\Sigma_g) - 3N_2(\Sigma_g) &= 0. \end{aligned} \tag{31}$$

Hence, the critical value is given by

$$R_{crit} = \frac{8\pi}{\sqrt{3}} \frac{1}{N_2(\Sigma_g)} (2-2g). \tag{32}$$

Thus, we have the result

$$\begin{aligned} S^2, &\qquad R_{crit} > 0, \\ T^2, &\qquad R_{crit} = 0, \\ \Sigma_g, \ g > 1, &\qquad R_{crit} < 0. \end{aligned} \tag{33}$$

4 Four Dimensions

Turning now to four dimensions, we begin with some specific topologies before considering some general classes. In [17], triangulations of $S^2 \times S^1$ and T^3 have been constructed. As we have seen, the only information we need in order to determine the critical length is the number of simplices. We have [17]

$$\begin{aligned} N_0(S^2 \times S^1) &= 10, \\ N_1(S^2 \times S^1) &= 40, \\ N_2(S^2 \times S^1) &= 60, \\ N_3(S^2 \times S^1) &= 30, \end{aligned} \tag{34}$$

and

$$\begin{aligned} N_0(T^3) &= 15, \\ N_1(T^3) &= 105, \\ N_2(T^3) &= 180, \\ N_3(T^3) &= 90. \end{aligned} \tag{35}$$

From (26), we then find that

$$\begin{aligned} S^2 \times S^1, &\qquad R_{crit} > 0, \\ T^3, &\qquad R_{crit} < 0. \end{aligned} \tag{36}$$

4.1 Tower Construction of $\Sigma_g \times S^1$

We are interested here in the construction of manifolds which have a product structure $\Sigma_g \times S^1$. Let us suppose we have a simplicial complex $K(\Sigma_g)$. To construct a simplicial manifold of the form $\Sigma_g \times S^1$, we can proceed as follows. For each 2-simplex of $K(\Sigma_g)$, we form a tower of 3-simplices containing nine elements. For example, let $[1,2,3]$ be a 2-simplex of $K(\Sigma_g)$ which has positive orientation. Then the associated tower is given by

$$\begin{aligned} &+ \; [1,2,3,\tilde{1}] \\ &+ \; [2,3,\tilde{1},\tilde{2}] \\ &+ \; [3,\tilde{1},\tilde{2},\tilde{3}] \\ &+ \; [\tilde{1},\tilde{2},\tilde{3},\hat{1}] \\ &+ \; [\tilde{2},\tilde{3},\hat{1},\hat{2}] \\ &+ \; [\tilde{3},\hat{1},\hat{2},\hat{3}] \\ &+ \; [\hat{1},\hat{2},\hat{3},1] \\ &+ \; [\hat{2},\hat{3},1,2] \\ &+ \; [\hat{3},1,2,3], \end{aligned} \tag{37}$$

where we have introduced six new vertices $\tilde{1},\tilde{2},\tilde{3}$ and $\hat{1},\hat{2},\hat{3}$, and the relative orientations are as shown. Thus, the number of 0-simplices and 3-simplices of the simplicial manifold

$K(\Sigma_g \times S^1)$ are

$$\begin{aligned} N_0(\Sigma_g \times S^1) &= 3N_0(\Sigma_g), \\ N_3(\Sigma_g \times S^1) &= 9N_2(\Sigma_g). \end{aligned} \tag{38}$$

However, the three-dimensional simplicial complex constructed in this way is a simplicial manifold, and so we have the Dehn-Sommerville relations

$$\begin{aligned} N_0(\Sigma_g \times S^1) - N_1(\Sigma_g \times S^1) + N_2(\Sigma_g \times S^1) - N_3(\Sigma_g \times S^1) &= 0, \\ N_2(\Sigma_g \times S^1) - 2N_3(\Sigma_g \times S^1) &= 0. \end{aligned} \tag{39}$$

Using (38), these relations provide the following information

$$\begin{aligned} N_2(\Sigma_g \times S^1) &= 18N_2(\Sigma_g), \\ N_1(\Sigma_g \times S^1) &= 9N_2(\Sigma_g) + 3N_0(\Sigma_g). \end{aligned} \tag{40}$$

Thus, we have determined the number of simplices $N_i(\Sigma_g \times S^1)$ for $i = 0, 1, 2, 3$ in terms of data associated with Σ_g. This can be further simplified by recalling that Σ_g is itself a simplicial manifold, and hence satisfies the Dehn-Sommerville relations (31). We can therefore express $N_i(\Sigma_g \times S^1)$ in terms of the number of vertices of Σ_g as follows

$$\begin{aligned} N_0(\Sigma_g \times S^1) &= 3N_0(\Sigma_g), \\ N_1(\Sigma_g \times S^1) &= 21N_0(\Sigma_g) - 18(2 - 2g), \\ N_2(\Sigma_g \times S^1) &= 36N_0(\Sigma_g) - 36(2 - 2g), \\ N_3(\Sigma_g \times S^1) &= 18N_0(\Sigma_g) - 18(2 - 2g). \end{aligned} \tag{41}$$

It has been shown that the number of vertices of a triangulation of a genus g Riemann surface satisfies the inequality [17]

$$N_0(\Sigma_g) \geq \frac{1}{2}\left(7 + \sqrt{49 - 24(2 - 2g)}\right). \tag{42}$$

It is then straightforward to evaluate the critical length, yielding the result

$$\begin{aligned} S^2 \times S^1, &\qquad R_{crit} > 0, \\ \Sigma_g \times S^1,\ g \geq 1 &\qquad R_{crit} < 0. \end{aligned} \tag{43}$$

4.2 Construction of a Connected Sum Manifold

A second general class of topologies which can be studied is provided by the connected sum structure. Let us consider two simplicial manifolds M_1 and M_2. We can construct a simplicial manifold called the connected sum of M_1 and M_2 and denoted $M_1 \# M_2$ as follows. We remove a single 3-simplex, say $[0, 1, 2, 3]$, from M_1. We recall that a 3-simplex is a 3-ball with boundary S^2, i.e.,

$$\partial[0, 1, 2, 3] = [1, 2, 3] - [0, 2, 3] + [0, 1, 3] - [0, 1, 2] = S^2. \tag{44}$$

Thus, upon removal of a single 3-simplex, we obtain a simplicial complex which we will denote by M_1', and with S^2 boundary $\partial[0, 1, 2, 3]$. We also remove a single 3-simplex, say

$[\tilde{0},\tilde{1},\tilde{2},\tilde{3}]$, from M_2, which results in a simplicial complex denoted by M_2', and with S^2 boundary $\partial[\tilde{0},\tilde{1},\tilde{2},\tilde{3}]$. To obtain a simplicial complex for $M_1\#M_2$, we must now identify the two S^2's. However, in order to ensure that the resulting structure is a simplicial manifold, this identification is performed with the aid of a tubular neighbourhood. A tubular neighbourhood is a simplicial complex which has the topology of a cylinder $S^2\times I$, where I is the unit interval. To construct such an object, we simply take a simplicial complex for S^2, and use the tower construction described above, except that now we need only three layers in the tower, viz.

$$\begin{aligned} K(S^2\times I) &= [1,2,3,\tilde{1}] - [0,2,3,\tilde{0}] + [0,1,3,\tilde{0}] - [0,1,2,\tilde{0}] \\ &+ [2,3,\tilde{1},\tilde{2}] - [2,3,\tilde{0},\tilde{2}] + [1,3,\tilde{0},\tilde{1}] - [1,2,\tilde{0},\tilde{1}] \\ &+ [3,\tilde{1},\tilde{2},\tilde{3}] - [3,\tilde{0},\tilde{2},\tilde{3}] + [3,\tilde{0},\tilde{1},\tilde{3}] - [2,\tilde{0},\tilde{1},\tilde{2}]. \end{aligned} \tag{45}$$

It is simple to check that

$$\partial K(S^2\times I) = -\partial[0,1,2,3] + \partial[\tilde{0},\tilde{1},\tilde{2},\tilde{3}]. \tag{46}$$

Thus, $K(S^2\times I)$ is a simplicial complex with boundary given by the disjoint union of two S^2's. To obtain the simplicial manifold $M_1\#M_2$, we now identify the S^2 boundaries of M_1' and M_2' with the boundaries of $S^2\times I$.

Since our goal is to compute the critical length, we must determine $N_1(M_1\#M_2)$ and $N_3(M_1\#M_2)$. To this end, we first note that from the explicit construction (45) we have

$$\begin{aligned} N_0(S^2\times I) &= 8, \\ N_1(S^2\times I) &= 22, \\ N_2(S^2\times I) &= 28, \\ N_3(S^2\times I) &= 12. \end{aligned} \tag{47}$$

This leads to the following result

$$\begin{aligned} N_0(M_1\#M_2) &= N_0(M_1) + N_0(M_2), \\ N_1(M_1\#M_2) &= N_1(M_1) + N_1(M_2) + 10, \\ N_2(M_1\#M_2) &= 2N_3(M_1) + 2N_3(M_2) + 20, \\ N_3(M_1\#M_2) &= N_3(M_1) + N_3(M_2) + 10. \end{aligned} \tag{48}$$

The number of 3-simplices of the connected sum is easily seen to be given by (48) by recalling that we have removed two 3-simplices in the construction of M_1' and M_2', and added twelve through the tubular neighbourhood. Similarly, one notes that the tubular neighbourhood involves the addition of ten extra 1-simplices not already present in either M_1' or M_2'. With this information at hand, we can now proceed and check the value of R_{crit} for several cases. Using the tower construction of the previous section, we obtain the following result

$$\begin{aligned} (S^2\times S^1)\#(S^2\times S^1), &\qquad R_{crit} > 0, \\ (S^2\times S^1)\#T^3, &\qquad R_{crit} > 0, \\ (S^2\times S^1)\#(\Sigma_g\times S^1), &\qquad R_{crit} < 0, \;\text{ for }\; g\geq 2, \\ (\Sigma_{g_1}\times S^1)\#(\Sigma_{g_2}\times S^1), &\qquad R_{crit} < 0, \;\text{ for }\; g_1\geq 1, \;\; g_2\geq 1. \end{aligned} \tag{49}$$

In [10], the wave function for a lens space boundary was considered. A simplicial complex for a lens space of the type $L(k,1)$ with $k \geq 2$ has been constructed in [18], with the data

$$\begin{aligned} N_0(L(k,1)) &= 2k+7, \\ N_1(L(k,1)) &= 2k^2+12k+19, \\ N_2(L(k,1)) &= 4k^2+20k+24, \\ N_3(L(k,1)) &= 2k^2+10k+12. \end{aligned} \tag{50}$$

We then find the critical lengths

$$\begin{array}{lll} L(k_1,1)\#L(k_2,1), & R_{crit} > 0, & \text{for } (k_1,k_2) = (2,2),(2,3),(2,4),(3,3),(3,4), \\ L(k_1,1)\#L(k_2,1), & R_{crit} < 0, & \text{otherwise.} \end{array} \tag{51}$$

Finally, using the tower construction of the product manifold, we find

$$\begin{array}{rll} L(k,1)\#(S^2\times S^1), & R_{crit} > 0, & \text{for } k = 2,3,4,5, \\ L(k,1)\#(S^2\times S^1), & R_{crit} < 0, & \text{for } k \geq 6, \\ L(k,1)\#T^3, & R_{crit} > 0, & \text{for } k = 2, \\ L(k,1)\#T^3, & R_{crit} < 0, & \text{for } k \geq 3, \\ L(k,1)\#(\Sigma_g\times S^1), & R_{crit} < 0, & \text{for } k \geq 2 \text{ and } g \geq 2. \end{array} \tag{52}$$

5 Five Dimensions

A general construction of a simplicial complex for $S^{d-1}\times S^1$ and T^d has been presented in [17]. For the case of $d=4$, we have

$$\begin{aligned} N_0(S^3\times S^1) &= 11, \\ N_1(S^3\times S^1) &= 55, \\ N_2(S^3\times S^1) &= 110, \\ N_3(S^3\times S^1) &= 110, \\ N_4(S^3\times S^1) &= 44, \end{aligned} \tag{53}$$

and

$$\begin{aligned} N_0(T^4) &= 31, \\ N_1(T^4) &= 465, \\ N_2(T^4) &= 1550, \\ N_3(T^4) &= 1860, \\ N_4(T^4) &= 744. \end{aligned} \tag{54}$$

This leads to the result

$$\begin{array}{rl} S^3\times S^1, & R_{crit} > 0, \\ T^4, & R_{crit} < 0. \end{array} \tag{55}$$

Finally, it is interesting to study the behaviour of the critical length for a series of triangulations of CP^2 constructed in [19]. For each $p \geq 2$, there is a triangulation of CP^2 with the following data

$$\begin{aligned}
N_0(CP^2_p) &= p^2+p+4, \\
N_1(CP^2_p) &= 3p(p^2+p+1), \\
N_2(CP^2_p) &= 2(6p-5)(p^2+p+1), \\
N_3(CP^2_p) &= 15(p-1)(p^2+p+1), \\
N_4(CP^2_p) &= 6(p-1)(p^2+p+1).
\end{aligned} \tag{56}$$

We find that $R_{crit} > 0$ for $p = 2, 3, 4$, and $R_{crit} < 0$ for $p \geq 5$. A triangulation of CP^2 with nine vertices has also been constructed [20], such that

$$\begin{aligned}
N_0(CP^2_9) &= 9, \\
N_1(CP^2_9) &= 36, \\
N_2(CP^2_9) &= 84, \\
N_3(CP^2_9) &= 90, \\
N_4(CP^2_9) &= 36.
\end{aligned} \tag{57}$$

The result is that $R_{crit}(CP^2_9) > 0$.

6 d Dimensions

While we have discussed some examples in specific dimensions in the preceding sections, we can now deal with a general example. The original model discussed in [7] dealt with the case of a universe with S^3 topology. It was found that R_{crit} was positive. Furthermore, for $R < R_{crit}$ the wave function was exponential in form and dominated semiclassically by real geometry of Euclidean signature. For $R > R_{crit}$, the wave function was oscillatory and dominated semiclassically by real geometry of Lorentzian signature. It is quite straightforward to extend this analysis to arbitrary dimensions. A simplicial complex for S^{d-1} is easily obtained as the boundary of a d-simplex, namely

$$K(S^{d-1}) = \partial[0, 1, 2, ..., d]. \tag{58}$$

A simplicial manifold is said to be k-neighbourly if the following condition is satisfied [17]

$$N_{k-1}(M) = \begin{pmatrix} N_0(M) \\ k \end{pmatrix}. \tag{59}$$

Using (58), one then sees that S^{d-1} is k-neighbourly for all k. Thus, we find that

$$\begin{aligned}
N_{d-1}(S^{d-1}) &= (d+1), \\
N_{d-3}(S^{d-1}) &= \frac{(d+1)d(d-1)}{3!}.
\end{aligned} \tag{60}$$

As a result, we determine that $R_{crit} > 0$ for all d. The behaviour of the wave function is also such that for $R < R_{crit}$ and $R > R_{crit}$, it is dominated semiclassically by real geometry of Euclidean and Lorentzian signature, respectively.

7 Concluding Remarks

We have shown that the analytic structure of the Regge action on a cone in d dimensions in simplicial minisuperspace can be obtained explicitly. This structure allows to us determine the form of the wave function, and we have shown that the closed contour of integration found in four dimensions [7] is equally valid in this more general setting. The wave function depends crucially on the critical value for the boundary edge length. For values of $R < R_{crit}$, the wave function is exponential in form, and is dominated in the semiclassical regime by real simplicial geometries of Euclidean signature. For values $R > R_{crit}$, the wave function has an oscillating form, and is dominated semiclassically by real simplicial geometries of Lorentzian signature. We should also note that contours of integration in continuum minisuperspace models have been studied in [15] and [21]-[23]. A calculation in three-dimensional Regge calculus with torus topology was presented in [24]. The original model studied in [7] was generalized in [25] to include anisotropy of the bounding universe.

Finally, we note that the behaviour of the wave function under subdivision of the bounding universe in four dimensions was studied in [10]. Such an analysis can also be performed in d dimensions. We simply have to appeal to the general (k, l) moves [26], and determine their effect on the value of R_{crit}.

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